AMATYC STUDENT MATH LEAGUE
1991-92 Solution Summaries

Test #1

E 1. $(\log 8)/(\log 4) = (3 \log 2)/(2 \log 2) = 3/2$.

E 2. $1/\cos x = \sin x/\cos x$, which only occurs if $\sin x = 1$ or $x = \pi/2$, but tan and sec are undefined there.

B 3. Squaring both sides yields $2 + \sqrt{x} = x$, or $x^2 - 5x + 4 = 0$. Of the solutions 4 and 1, only 4 checks.

D 4. $\log_{16}(1/2) = -1/4$, not -4.

B 5. There are ${}_6C_3 = (6 \cdot 5 \cdot 4)/(3 \cdot 2 \cdot 1) = 20$ possible committees. There are 3 ways to choose 2 out of 3 men, 3 ways to choose 1 out of 2 women, so the required probability is $(3 \cdot 3)/20 = 0.45$.

A 6. The inequality is equivalent to $3x \geq -12$ or $x \geq -4$, which defines the interval $[-4,+\infty)$.

C 7. Clearly, if there were 3 or more students in the room, we could find a group of 3 people with no non-students. With 1 or 2 students in the room, all conditions are satisfied.

D or E 8. $h(h(x)) = 1/((1/x) + 1) + 1 = 1 + x/(x + 1)$; E is correct if we require that $x \neq 0$ must be stated.

C 9. If V = total price, p = Al's price per donut, then $V = 5(p+4)$ and $V = 6p$, so $5p+20 = 6p$. Then $p = 20$ and V = \$1.20. Then Bo pays \$1.20/8 = 15¢ per donut , which is 5¢ less than p.

B 10. Let A, B, C represent the three radii. Then $A+B = 5$, $B+C = 7$. $A+C = 8$. Subtracting the first two equations gives $A-C = -2$, adding the third gives $2A = 6$, so $A = 3$.

11. NOT GRADED (statement was ambiguous as to which camps were on the river).

C 12. The # of decimal digits in a number N is the next integer above log N (try it). $\log 2^{50} = 50 \log 2$, and log 2 is slightly bigger than 0.3 (since $2^{10} = 1{,}024$ and $10^3 = 1{,}000$), so $50 \log 2 \approx 15$; next integer is 16.

D 13. In standard form $x^2/16 + y^2/25 = 1$, so $a^2 = 25$, $b^2 = 16$, and $c^2 = a^2 - b^2 = 9$, so $c = 3$. Since the quantity under y is greater, the y-axis is the major axis and the foci are at $(0, \pm 3)$.

A 14. From the sum-difference formulas we have $\cos(A-B) + \cos(A+B) = 2 \cos A \cos B$, $\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$. Letting $A = 5x$, $B = x$ we get $(2 \sin 5x \sin x)/(2 \cos 5x \cos x) = \tan 5x/\tan x$.

A 15. Letting each letter A,M,T,Y,C appear at most once gives $5 \cdot 4 \cdot 3 \cdot 2 = 120$ words. Selecting 2 A's gives 6 ways to place the A's, $4 \cdot 3$ ways to place the other 2 letters for $6 \cdot 4 \cdot 3 = 72$ more words; total = 192.

B 16. If k=1, the system is consistent, since all three equations are then the same, so assume $k \neq 1$. Subtracting Eq2 from Eq3 and adding -k times Eq2 to Eq1 gives $(1-k^2)y + (1-k)z = 2-2k$, $(1-k)y + (k-1)z = 0$. Dividing out 1-k (OK, since $k \neq 1$) gives $y - z = 0$, $(1+k)y + z = 2$; adding $-(1+k)$ times the first equation to the second gives $(2+k)z = 2$, which is consistent unless $k = -2$, so only one value fails.

E 17. By the addition formula for tangent, this = $\tan\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) = \tan\frac{4\pi}{8} = \tan\frac{\pi}{2}$, which is undefined.

C 18. The graph is a right rectangular hyperbola with horizontal asymptote $y = 2$, so y is any value except 2.

C 19. The horizontal asymptote is at $y = 1$, and $1 = (x-1)^2(x+6)/x^3$ yields $4x^2-11x+6 = 0$, with 2 solutions.

12 20. Let r = radius of circle M. Then $MQ = r+8$, $MP = r$, $PQ = 2r-8$, and $MQ^2 + PQ^2 = MQ^2$. Thus $(r+8)^2 = r^2 + (2r-8)^2$ or $4r(r-12) = 0$. Since $r \neq 0$, r must be 12.

Test #2

C 1. P must have coordinates (3,2) (symmetry to y=x reverses coordinates), so Q has coordinates (-3,2).

A 2. $\sqrt{18} + \sqrt{32} = \sqrt{9}\sqrt{2} + \sqrt{16}\sqrt{2} = 3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}$.

E 3. $1992 = 2^3 \cdot 249 = 2^3 \cdot 3 \cdot 83$, all prime. Then $83 = 10a+b$, so $a = 8$, $b=3$, and 8, 3, $8^{1/3} = 2$ are all factors.

C 4. Since $(x^2)^2 = x^4$, the required term is ${}_7C_2(x^2)^2(-2)^5 = 21(-32)x^4 = -672x^4$.

C 5. $(x+1)^2 - x^2 = x^2+2x+1 - x^2 = 2x+1$, and $(2x+1)/2 = 2x/2 + 1/2 = x + 1/2$.

C 6. Cubing both sides gives $1 + \sqrt{x} + 3\sqrt[3]{1 + \sqrt{x}}\sqrt[3]{1 + \sqrt{x}}\sqrt[3]{1 - \sqrt{x}} + 3\sqrt[3]{1 + \sqrt{x}}\sqrt[3]{1 - \sqrt{x}}$

$\sqrt[3]{1-\sqrt{x}} + 1 - \sqrt{x} = 1$, or $3\sqrt[3]{1+\sqrt{x}}\,\sqrt[3]{1-x} + 3\sqrt[3]{1-x}\,\sqrt[3]{1-\sqrt{x}} = 3$, which gives $\sqrt[3]{1-x}\left(\sqrt[3]{1+\sqrt{x}} + \sqrt[3]{1-\sqrt{x}}\right) = 1$. But the expression in parenthesis is the left side of the original equation, so $\sqrt[3]{1-x}\left(\sqrt[3]{5}\right) = 1$, $5(1-x) = 1$, and $x = 4/5$.

A 7. Let G be the point where $\overline{DE}$ is tangent to circle F. Then DG = DB, EG = EC, so the perimeter of $\Delta ADE = AB + AC = 2AB = 16$, so AB = 8. The Pythagorean Theorem yields $AF = \sqrt{8^2 + 4^2} = 4\sqrt{5}$.

B 8. The equation for exponential growth is $A = A_0 2^{t/k}$, where k is the doubling time. Thus the equation is $A = (20000)2^{t/2}$. Letting A = 100,000 yields $t = 2\log_2 5$. Since $2^{9/4} = 4\sqrt[4]{2} \approx 4.8$, $2\log_2 5$ is slightly more than 4.5, so the population reaches 100,000 shortly after Jan. 1, 1994.

A 9. $\csc x - \cot x = (1 - \cos x)/\sin x$; $\sin x > 0$ on $(0,\pi)$; $1 > \cos x$, so $1 - \cos x > 0$ on $(0,\pi)$.

E 10. Let I = income, x = number of posters over 10. $I = (5 - 0.15x)(10+x) = 50\frac{7}{12} - 0.15\left(x - 11\frac{2}{3}\right)^2$. Max. is at $x = 11\frac{2}{3}$, but number of posters must be an integer. For x = 11, I = \$70.35, for x = 12, I = \$70.40, so maximum income occurs when 10 + x = 22 posters are put up.

D 11. $x^6 = 64$ means $x^6 - 64 = (x^3-8)(x^3+8) = (x-2)(x^2+2x+4)(x+2)(x^2-2x+4) = 0$. The product of the imaginary roots is then $(1+i\sqrt{3})(1-i\sqrt{3})(-1+i\sqrt{3})(-1-i\sqrt{3}) = (1-3i^2)(1-3i^2) = (1+3)(1+3) = 16$.

B 12. u=5/3, v = -5, so the new equation (with slope 1/3) is y = (1/3)x + (5/3) or 3y - x = 5.

C 13. AMA = 5(TYC), so T=1 (T≥2 produces a 4-digit result). Also, A=0 or 5. Checking 505, 515, etc and eliminating contradictory results yields 535 = 5(107) and 545 = 5(109), so M = 3 or 4.

C 14. The Remainder Theorem says the remainder is P(-1) = 5 + 3 - 1 + 4 - 2 = 9.

A 15. Check the intervals $(-\infty,-2),(-2,1),(1,3/2),(3/2,+\infty)$. The resulting equations are -3x+4 = -x-2, -3x+4 = x+2, -x+2 = x+2, 3x-4 = x+2 with respective solutions 3, 1/2, 0, 3. Only 1/2 and 3 check.

D 16. John could choose 3 possible electives at 10, leaving 4x3 = 12 choices for 11 and 12. Similarly, he has 3x12 choices with an 11:00 elective and 36 more with a 12:00 elective, for 3(36) = 108 choices.

A 17. Let the perpendicular distances from P to AB, BC, CD, DA = a, b, c, d respectively. Then $196 = a^2+d^2$, $4 = a^2+b^2$, $64 = b^2+c^2$, $PD^2 = c^2+d^2$, so $192 = d^2-b^2 = PD^2-64$, $PD^2 = 256$, and PD = 16.

A or C 18. From the graph it is clear the two curves do not intersect in the real plane. However, equating the expressions $(x^2-4)/(x+2)$ and $4/(3-x)$ yields x2-5x+10 = 0, which has two complex solutions.

B 19. The terms must be a-1, a+4-2, a+8-2, so $a+2 = r(a-1)$, $a+6 = r^2(a-1)$. Thus $(a+2)^2 = r^2(a-1)^2 = (a+6)(a-1)$, or a = 10. Then the arithmetic sequence is 10, 14, 18, the geometric is 9, 12, 16, and r = 4/3.

E 20. $\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\frac{1}{2} - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{2} - \sqrt{6}}{4}$, which is not any of the expressions.

Test #3

E 1. Angles on the same side of a transversal are supplementary, so m∠YCT = 80°. AYCM is an isosceles trapezoid and AMT an isosceles triangle, so m∠AMT = m∠YCT = m∠MTA, and m∠MAT = 180°-160°.

A 2. k odd produces k regions, k even produces 2k regions, so 7, 8, 9 are possible, 6 is not.

D 3. Solving 3x+2y = 6, 6x+2y = 13 yields the point (7/3,-1/2), which is in the fourth quadrant.

D 4. Substitute y = 2/x in the first equation to get $x^4 - 20x^2 + 4 = 0$. Then $x^2 = 10 \pm 4\sqrt{6}$, so $x = 2 \pm \sqrt{6}$ or $-2 \pm \sqrt{6}$. Adding the corresponding y-values gives $2\sqrt{6}$, $-2\sqrt{6}$, $2\sqrt{6}$, and $-2\sqrt{6}$, with product 576.

C 5. 8 = 2C - 0.5W, 20 = C+B+W. Substitution gives 5C+B = 36. Eliminating possibilities for which any of C, B, or W are negative or over 20 gives (C,B,W) = (4,16,0),(5,11,4),(6,6,8),(7,1,12).

D 6. log (k/(k+1)) = log k - log(k+1), so all terms cancel except log 1 - log 100 = 0 - 2 = -2.

C 7. After n years, A's salary is 10000+(n-1)1000, while B's salary is 10000+(4n-3)x, where x is B's (semiannual) raise. We want $10000+(4n-3)x \geq 10000+(n-1)1000$ for all n, or $x \geq 1000((n-1)/(4n-3))$. Now $(n-1)/(4n-3) < 1/4$ for all n, but approaches 1/4 for large n, so we must have $x \geq 1000(1/4) = \$250$.

A 8. Since Ed and Al contradict each other, one must be lying, and Jo and Cy tell the truth. Thus Ed is lying, Al is telling the truth, and Ed must be second tallest (order is Al, Ed, Jo, Cy).

B 9. $\cot \frac{1}{2} x = \frac{\cos(1/2)x}{\sin(1/2)x} = \frac{(2\cos^2(1/2)x - 1)+1}{2\sin(1/2)x\cos(1/2)x} = \frac{\cos x + 1}{\sin x}$. (Note: choice A = $-\cot \frac{1}{2} x$).

D 10. $4^{3/2} = 8$, $16^{3/2} = 64$, so $\log_4 8 + \log_{16} 64 = \log_4 8 + \log_4 8 = \log_4(8 \cdot 8) = \log_4 64$.

D 11. $\angle AEB$ is an exterior angle of ΔDEB, so $m\angle AEB = m\angle EBD + m\angle ADB = 2(m\angle ADB)$, so $m\angle EBD = m\angle ADB$ and DE = BE. Since $AB^2 + AE^2 = BE^2$, then $AB^2 + AE^2 = DE^2$.

B 12. For $f(x) = x^2$, $f(1+2) = 9 \neq f(1)+f(2) = 5$; for $f(x) = x/2$, $f(a+b) = (a+b)/2 = a/2 + b/2 = f(a)+f(b)$; for $f(x) = 2^x$, $f(1+2) = 8 \neq f(1)+f(2) = 6$.

C 13. $2\sin x - \cos^2 x = 2\sin x - (1 - \sin^2 x) = \sin^2 x + 2\sin x - 1 = \sin^2 x + 2\sin x + 1) - 2 = (\sin x + 1)^2 - 2$. Since a squared quantity is always ≥ 0, the expression must have a minimum of 0 - 2 or -2.

D 14. $x^3+y^3 = 1$ means $y = (1-x^3)^{1/3}$. For large x, $1-x^3 \approx -x^3$, so $y \approx (-x^3)^{1/3} = -x$ and $x + y \approx 0$.

A 15. 1, 4, 6 cannot stand alone (they're not prime), and 4 and 6 can't be final digits (the numbers would be divisible by 2). Thus the set must include numbers of the form 4X and 6X. 6X cannot end in 2, 3, or 5 (none of those are prime), so it must be 61 or 67. 4X can't end in 2 or 5, so it is 41, 43, or 47. Thus possible solutions are {2,3,5,47,61}, {2,3,5,41,67},{2,5,7,43,61} (all adding to 118), none of which contain 23. (Sneaky argument: replacing 23 with 2, 3 would always reduce the total, so 23 can't be in it).

E 16. If the 30° angle is between the 2 given sides, we have a unique (SAS) triangle, solvable by the Law of Cosines. If the 30° angle is at the end of the side of length 6, we have a unique triangle solvable by the Law of Sines. If the angle is at the end of the side of length 8, we have the ambiguous case of the Law of Sines and two distinct riangles, for a total of 4 noncongruent possible triangles. For an exercise, students may wish to solve all three possibilities to convince themselves that there are indeed four triangles.

17. NOT GRADED. If $r \neq s$, then $a_r = s$, $a_s = r$ implies $a+(r-1)d = s$, $a+(s-1)d = r$, so $a-d = s-rd = r-sd$ and $d(s-r) = r-s$. If $r \neq s$, we can divide by s-r to get $d = -1$, so $a+1 = r+s$. Then $a_{r+s} = a+(r+s-1)(-1) = a-(r+s)+1 = a+1 - (a+1) = 0$. If the restriction $r \neq s$ is omitted, any of the five given choices are possible (for extra credit, find an example for each of the five).

A 18. Total number of words = $4^4 = 256$. Number of words with exactly 3 different letters = (number of ways to choose duplicate letter)(number of ways to place 2 duplicate letters)(number of ways to choose remaining 2 letters) = $4 \cdot 6 \cdot (3 \cdot 2) = 144$, so probability = 144/256 = 9/16.

B 19. $xy = x/y$ means $xy^2 = x$ or $x(y^2 - 1) = 0$, so x=0 or $y = \pm 1$. But x=0 means $0y = 0+y$ or y=0, impossible since x/y is defined. Thus $y = \pm 1$. But y=1 means $x = x+1$ (impossible), so $y = -1$, $-x = x-1$, and $x = 1/2$. Then $x - y = 1/2 - (-1) = 3/2$.

B 20. Cards are reversed for every integral factor they possess, so a card with an even number of factors ends up red, an odd number of factors ends up blue. For every factor below the square root of N, N has a factor above the square root of N (a pair), so the only numbers with an odd number of factors are perfect squares. There are 14 perfect squares from 1 to 200 ($1^2, 2^2, \ldots, 14^2 = 196$), so there are 200 - 14 = 186 red cards at the end.