Test #3 April 1994

1 D $12^8 = (2^2 \cdot 3)^8 = 2^{16}3^8$. Divisors have the form $2^m 3^n$, where there are 17 possibilities for m and 9 possibilities for n. Thus, there are 17 • 9=153 divisors.

2 A $\frac{3}{14} = 0.2\overline{142857}$. The (2+6k)th dec digit is 1. The (3+6k)th dec digit is 4. Etc.
$\frac{1994}{6} = 332 + \frac{2}{6}$. The (2+6 • 332)th dec digit is 1.

3 E $y = \frac{(x-2)^2}{x+1} = \frac{x^2-4x+4}{x+1} = x-5+\frac{9}{x+1}$. There is one vertical ($x=-1$) and one oblique ($y = x-5$) asymptote.

4 C $\frac{37\pi}{7} = \left(5+\frac{2}{7}\right)\pi$, a 3rd quadrant angle with related angle $\frac{2\pi}{7}$.
Thus, $\cos\frac{37\pi}{7} < 0$. This means $\frac{\pi}{2} < \text{Cos}^{-1}\left(\cos\frac{37\pi}{7}\right) < \pi$.
Since the related angle is $\frac{2\pi}{7}$, the answer must be $\pi - \frac{2\pi}{7} = \frac{5\pi}{7}$.

5 D The only possibility is that the mothers bore 4, 5, 6, and 8 kittens.

6 D a) and b) have centers on the x-axis with large radii. c) passes through the origin. e) has center (6,1) with radius > 3. Certainly, $(2\pi,0)$ is within this circle.
d) has center $\left(-\frac{3}{2},\frac{3}{2}\right)$ with radius $\frac{\sqrt{6}}{2}$. This circle is entirely above the x-axis and between the vertical lines $x = -3$ and $x = 0$. But $\sin x < 0$ for these values of x.

7 B $\log_2 x + \log_2(x^2+2x-1) = 1 \Rightarrow \log_2(x^3+2x^2-x) = 1 \Rightarrow x^3+2x^2-x = 1 \Rightarrow x^3+2x^2-x-1 = 0$
$\Rightarrow (x-1)(x+1)(x+2) = 0 \Rightarrow x = -1, 1, -2$. -1 and -2 are extraneous. Only 1 checks.

8 D $\frac{1}{5}\left(x+\frac{2}{5}\right) = \frac{4}{5}x - \frac{3}{5} \Rightarrow x + \frac{2}{5} = 4x-3 \Rightarrow 3x = \frac{17}{5} \Rightarrow x = \frac{17}{15}$
$\frac{2}{5}x = \frac{2}{5}\cdot\frac{17}{15} = \frac{34}{75}$

9 B Either consider four cases: $x \le -5/2$, $-5/2 < x \le -1$, $-1 < x \le 0$, and $x > 0$ or sketch the graphs of the left and right hand sides.

10 A The only counting numbers with exactly 3 positive divisors have the form p^2 where p is prime. (The divisors are $1, p, p^2$). The primes p with $p^2 < 1000$ are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31.

11 E $x^2 = \sqrt{5x^2-4} \Rightarrow x^4 = 5x^2 - 4 \Rightarrow x^4 - 5x^2 + 4 = 0 \Rightarrow (x^2-1)(x^2-4) = 0 \Rightarrow x = \pm 1, x = \pm 2$
All four check. $(-2)^2 + (2)^2 + (-1)^2 + (1)^2 = 4+4+1+1 = 10$

12 C If BC < 5 there are no triangles. If 5 < BC < 10 there are two triangles. If BC ≥ 10 or BC = 5 there is one triangle. Thus, M = 0, N = 2, and P = 1. M + 2N + 3P = 7.

13 D Note that $0 < \frac{10}{x^2+1} < 10$. Thus $\frac{10}{x^2+1} = \pi, 2\pi, 3\pi$. Each choice leads to 2 solutions.

14	E	Completing the squares yields $\frac{(y-2)^2}{9}-\frac{(x-2)^2}{16}=1$. We have a hyperbola with center at (2,2). $c^2=9+16=25 \Rightarrow c=5$. The foci are (2,7) and (2,-3) .
15	A	$\left(\frac{1+i}{\sqrt{2}}\right)^{20}=\left(\left(\frac{1+i}{\sqrt{2}}\right)^2\right)^{10}=\left(\frac{1+2i+(-1)}{2}\right)^{10}=(i)^{10}=-1$
16	C	The shortest distance occurs when the segment from A to her location D on the third leg of the journey is perpendicular to DC, where C is the location of the second turn. But AC = $800\sqrt{2}$, and $m\angle ACD=30°$. Thus AD = $\frac{1}{2}\cdot 800\sqrt{2}=400\sqrt{2}$
17	C	There are six possible rolls, and each corner occurs on exactly three of the six. Thus the sum of values of the six possible rolls is $3(1+2+3+4+5+6+7+8)=3(36)=108$. The mathematical expectation for one roll is $\frac{108}{6}=18$.
18	63	Let d = # dogs and c = # children. Then # legs = 4d + 2c, # eyes = 2d + 2c, # heads = d + c, and # tails = d . # legs = (1.90)(# eyes) and # heads = 7 + # tails. 4d + 2c = (1.9)(2d + 2c) and d + c = 7 + d. Thus c = 7 and 4d + 14 = 3.8d + 26.6. 0.2d = 12.6 . d = 63
19	369	10000_4 through 33333_4 (256 through 1023) Note: $33333_4=100000_4-1$ 1000_5 through 4444_5 (125 through 624) Note: $4444_5=10000_5-1$ 256 through 624 is 624-256+1 = 269 numbers.
20	$10/243$	At each intersection there are 3 possibilities. So after 5 rolls, there are $3^5=243$ possible paths. Of these, there are exactly 10 which have him at the starting point as he prepares for his sixth roll: NESWS, ENWSS, NWSES, WNESS, NESSW, NWSSE, ESSWN, WSSEN, EESWW, WWSEE.