Test #1 **Student Mathematics League Short Solutions** November 2001

A 1. Letting $t =$ *this* and $T =$ *that* yields the equations $(t + T)/3 = t^2$ and $T/t = 8/1$. Substituting $T = 8t$ into the first equation yields $3t^2 - 9t = 0$ or $3t(t - 3) = 0$, with solutions $t = 0,3$. But $t \neq 0$, so $t = 3$.

A 2. S is (2,3) and T is (-2,-3), so the required line has equation $y - 3 = \frac{3}{2}(x - 2)$, or $y = \frac{3}{2}x$.

E 3. The given inequality is equivalent to $-2001 \le 5 - 2x \le 2001$, or $1003 \ge x \ge -998$. The resulting interval, [-998,1003], is contained only in the interval [-1000,2001].

B 4. Let x be the given angle and y its supplement. $\sin x = \frac{2}{3}$ implies $\sin y = \frac{2}{3}$ and $\csc y = \frac{3}{2}$. Since $\cot^2 y + 1 = \csc^2 y$, then $\cot^2 y = \frac{5}{4}$ and $\cot y = \pm\frac{\sqrt{5}}{2}$. But y is in Quadrant II, so $\cot y = -\frac{\sqrt{5}}{2}$.

C 5. The given function values yield the equations $c = 3$, $4a + 2b + c = 0$, and $16a + 4b + c = 1$. This implies $4a + 2b = -3$ and $16a + 4b = -2$, so $4b = -10$ and $b = -2.5$. Then $a = 0.5$, so $a + b + c = 1$.

E 6. Since $Q(x) = \frac{(x-2)(x-1)}{x^2(x+1) - 4(x+1)} = \frac{(x-2)(x-1)}{(x-2)(x+2)(x+1)}$, the domain is all x except ±2 and -1.

B 7. Since $8^2 + 15^2 = 17^2$, the triangle is a right triangle. Any right triangle inscribed in a circle must be inscribed in a semicircle with the hypotenuse as the diameter. Thus the radius is half of 17 or 8.5.

D 8. The only 3-digit perfect cube with its first and last digits the same is 343, and the only 3-digit perfect squares divisible by 12 are 144, 324, 576, and 900, so TYC = 576. Thus A + M + A + T + Y + C = 28.

E 9. For $2x^2 - bx - 36$ to have rational solutions, its discriminant, $b^2 + 288$, must be a perfect square. This is true for $b = \pm1, \pm6, \pm14, \pm21, \pm34$, and $\pm71$, for a total of 12 values of b.

D 10. Since $\angle A = \angle CDA$, $\Delta ACD$ is isosceles with AC = CD. Then AB = 3 = AC/3, and since $\Delta ABF$ and $\Delta ACD$ are similar, BF = BE/3 = CD/3 = 3, and EF = BE - BF = 6.

D 11. At 4 mph, Matt takes (60 min)((1/2)/4) = 7.5 min to row to shore. This means (7.5 min)(10 gal/min) = 75 gal of water enter. Cassie must bail 45 gal (all but 30 of the 75 gal) in this 7.5 min, or 6 gal/min.

D 12. If r is the length of the radius of the circle, the radius ending at the arc's midpoint, the chord, and the radius ending at one end of the chord form a right triangle with legs of length r - 4 and 6 mm and hypotenuse r mm. Then $(r - 4)^2 + 6^2 = r^2$, so $r^2 - 8r + 52 = r^2$, $8r = 52$, and the radius is exactly 6.5 mm.

C 13. There are 5 ways to parenthesize: (a + b)*(c ^ d), ((a + b)* c) ^ d, (a + (b * c)) ^ d, a + ((b * c) ^ d), a + (b * (c ^ d)). Evaluating each expression yields 7, 49, 1, 19, and 7, for 4 distinct values.

B 14. We need the smallest N ending in 0, its digits adding to 8 or 17 (so that N + 1 is a multiple of 9), and N + 2 is a multiple of 8. The values satisfying the first two conditions are 80 + 90k. Since 350 is the smallest value for which N + 2 is a multiple of 8 (352 = 8x44), this is N. Only 11 is a factor of 352.

A 15. The line segments $\overline{PS}$, $\overline{TW}$, $\overline{QV}$, $\overline{RU}$ divide the octagon into 4 isosceles right triangles, 4 rectangles, and a square. If s is the length of a side of the octagon, the total area is $4(s^2/4) + 4(s^2/\sqrt{2}) + s^2 = (2 + 2\sqrt{2})s^2$. The area of PSTW $= 2(s^2/\sqrt{2}) + s^2 = (1 + \sqrt{2})s^2 =$ half the octagon's area.

E 16. The equation is equivalent to $x^3 + y^3 - x - y = 0$, or $(x + y)(x^2 - xy + y^2 - 1) = 0$. Thus $x + y = 0$ (a straight line), or $x^2 - xy + y^2 = 1$, an ellipse (since $B^2 - 4AC < 0$).

C 17. To maximize the product, the factors should be as nearly equal as possible. For n = 2, 3, 4, 5, 6, this yields 5(5), 3(3)(4), 2(2)(3)(3), 2(2)(2)(2)(2), and 1(1)(2)(2)(2)(2), or 25, 36, 36, 32, 16.

B 18. The number of ways to pick any 2 shoes is C(22,2) = 231. The number of ways to pick a matching pair is by picking 1 right and 1 left of the same color: 6(6) + 3(3) + 2(2) = 49. Then 49/231 = 7/33.

C 19. If A is (0,0) and B is (65,0), the lines thru A, C and B, E have equations $y = \frac{3}{4}x$ and $y = \frac{-5}{12}x + \frac{325}{12}$. They intersect at $\left(\frac{325}{14}, \frac{975}{56}\right)$, and the pentagon's area is $\frac{52(39)}{2} + \frac{25(60)}{2} - \frac{975}{56}\left(\frac{65}{2}\right) = 1198\frac{17}{112}$.

D 20. By symmetry they meet after 6 turns. Starting from (0,0) they end at (cos 0° + cos 15° + cos 30° + … + cos 75° + cos 90°, sin 0° + sin 15° + … + sin 90°), which is $4.30\sqrt{2}$ or about 6.1 from (0,0).