Test #3 **Student Mathematics League Short Solutions** March-April 2002

B 1. $\log_{36} 9 + \log_{36} 24 = \log_{36} 216$, so $216 = 36^x$, $6^3 = 6^{2x}$, and x = 1.5.

D 2. The angle sum in a regular 30-sided polygon is (30 - 2)180° = 5040°, so each angle is 5040/30 = 168°.

A 3. The first system's solution is (4,-1), so p = 4, q = -1. The other system's solution is (4,1), or (p,-q).

C 4. Since y = (c/360)(2πr) = (k/360)(2πR), then cr = kR, and c/k = R/r.

C 5. The angles are either k + 4k + 4k = 180, so k = 20° and 4k = 80°, or k + k + 4k = 180, so k = 30° and 4k = 120°. Thus the only choice which is not possible is 40°.

D 6. $\frac{2}{15} = \frac{1}{8} + \frac{1}{120} = \frac{1}{9} + \frac{1}{45} = \frac{1}{10} + \frac{1}{30} = \frac{1}{12} + \frac{1}{20}$. These are the only solutions for $8 \le a \le 14$, and thus no other solutions are possible, since 1/7 > 2/15, and 1/a must be more than half of 2/15.

D 7. The smallest value of k is at least GCF(36,21) = 3. Trial-and-error shows that (3,-5) is a solution when k = 3, and all solutions have the form (3 + 7i,-5 - 12i). Thus the smallest positive value for y is 7.

D 8. Since tangent and cotangent are cofunctions, cot x = tan (π/2 - x) = tan (-x + π/2), which is a reflection in the x-axis followed by a translation π/2 units to either the left or right (since period of tangent is π).

E 9. The x-intercepts of $T(x) = \cos x^{-2}$ on [0.05,1] are the same as those of $t(x) = T(x^{-0.5}) = \cos x$ on [1,400]. The x-intercepts of t(x) are at π/2 + πk and π/2 + 126π = 397.411, so the answer is 127.

E 10. Let h : m be the required time. The minute hand moves 6° per minute (360°/60), so its position is 6m. The hour hand starts at 30h (360°/12) and moves 1/2° per minute (30°/60), so its position is 30h + 0.5m. Then $|30h+0.5m-6m| = m/2$, so m = 5h or 6h. If m = 5h, h must be even to get a whole number angle, so the times are 2:10, 4:20, 6:30, 8:40, 10:50, 1:06, 2:12, 3:18, 4:24, 5:30, 6:36, 7:42, 8:48, and 9:54.

B 11. For AM · A to be 600 to 699, A must be 8 (A = 7 makes T ≤ 5, A = 9 makes T ≥ 8). Trying 80 up to 87 (88 and above makes T = 7) and avoiding repetition produces the unique solution 84 · 8 = 672.

B 12. A year is 52 weeks + 1 day, so dates shift 1 day later each year (2 in leap years), and 27 paychecks means getting one on Jan. 1. Juan gets paychecks on Jan. 6, 2003, Jan. 5, 2004, Jan. 3, 2005 (because 2004 is a leap year), Jan. 2, 2006, and Jan. 1, 2007, with 27 paychecks that year.

C 13. $M^2 = \begin{bmatrix} s^2 - 8 & 2s + 2v \\ -4s - 4v & v^2 - 8 \end{bmatrix}$, so $s = \pm\sqrt{8}$, v = -s, and $|s - v| = |2s| = 4\sqrt{2}$.

A 14. Since 2002 = 2(7)(11)(13), N = 2002. The product of 2002's factors is 2002^8, since the 16 factors can be paired up; but if P is the common product, the 16 factors multiply to P^4, so $P = 2002^2$. That means each row, column, and diagonal has 2 factors each of 2, 7, 11, and 13. By process of elimination, # = 22.

B 15. The allowed values for b & c are: 0:0,-1,-4,-9; ±1:0,-2,-6; ±2:1,0,-3,-8; ±3:2,0,-4,-10; ±4:4,3,0,-5; ±5:6,4, 0,-6; ±6:9,8,7,0,-7; ±7:10,6,0,-8; ±8:7,0,-9; ±9:8,0,-10; ±10:9,0. The answer is 76/441 ≈ 0.17.

E 16. Label the functions f1, f2, f3. Since f1(0) = f2(0), eliminate A; f1(1) = f2(1) = f3(1) eliminates B; f1(-1) = f3(-1) eliminates C. Solving f2(x) - f3(x) = 0 gives x = (c - a)/(b - c); this eliminates D.

A 17. The first several terms have the pattern $s_n^2 - s_{n-1}s_{n+1} = 5$ for odd n and -5 for even n. Assuming this pattern holds up to 2001, $s_{2002}^2 - s_{2001}s_{2003} = s_{2002}^2 - s_{2001}(s_{2001} + s_{2002}) = s_{2002}^2 - s_{2001}^2 - s_{2001}s_{2002} = s_{2002}^2 - (s_{2000}s_{2002} + 5) - s_{2001}s_{2002} = s_{2002}(s_{2002} - (s_{2000} + s_{2001})) - 5 = s_{2002}(0) - 5 = -5$.

B 18. For x ≤ -1/2, f(x) = -1; for -1/2 ≤ x ≤ 1/2, f(x) = 2x; for x ≥ 1/2, f(x) = 1. Thus f(f(x)) = 4x for -1/4 ≤ x ≤ 1/4, and f(f(f(x))) = 8x for -1/8 ≤ x ≤ 1/8. The required interval has length 1/8 - (-1/8) = 1/4.

A 19. Because $k(x) = (x - p)(x - q)(x - r) = x^3 - (p + q + r)x^2 + (pq + qr + pr)x - pqr$, then p + q + r = 5, pq + qr + pr = 4, and pqr = -8. But $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{qr + pr + pq}{pqr} = \frac{4}{-8}$ or $\frac{-1}{2}$.

A 20. By polynomial division, $P(x) = Q(x)(x^3 + x + 1) + (x - 2)$. Any root of both P(x) and Q(x) must also then be a root of x - 2. But the only root of x - 2 is 2, and $x^2 + ax + 1$ has a root of 2 only if a = -5/2.