

Test #2

Student Mathematics League Short Solutions January-February 2002

- E 1. $x^2 - 15x - 54 = (x - 18)(x + 3)$. No equation is satisfied by 18, and -3 satisfies only $6x + 15 = 2x + 3$.
- B 2. For an angle a , $180 - a = k(90 - a)$ or $a = \frac{90k - 180}{k - 1}$. If $k = 2$, $a = 0$, so $k > 2$. If $k = 3$, $a = 45^\circ$.
- D 3. Since hypotenuse PR in right $\triangle PQR$ is twice leg PQ, $\triangle PQR$ is a 30-60 right triangle with $\angle QRP = 30^\circ$ and $QR = \sqrt{3}$. The same argument shows $\angle SRP = 30^\circ$, so $\triangle SQR$ is equilateral and $SQ = QR = \sqrt{3}$.
- D 4. $16^{-2} = .00390625$ and $16^{-2.25} = .001953125$, so $\log_{16} .002$ is closer to -2.5 than any other choice.
- E 5. The parallelogram which is their intersection has base the hypotenuse of a 30-60 right triangle with longer leg 60 ft, and altitude 50 ft. The base is then $\frac{120}{\sqrt{3}}$ ft and the area is $\frac{120(50)}{\sqrt{3}}$ or about 3464 sq ft.
- A 6. The larger kitten must weigh half of 12 lb (since its weight equals the other 2), the smaller kitten must weigh a third of 12 lb (since its weight is half the other 2), so the puppy must weigh $12 - 6 - 4 = 2$ lb.
- B 7. If R is the final result, $(R - 2) + (R + 2) + (R/2) + 2R = 45$ and $R = 45(2/9) = 10$. The numbers are then 8, 12, 5, and 20 with a product of 9600.
- D 8. Since $2002 = 2(7)(11)(13)$, its positive integer factors are 1, 2, 7, 11, 13, 2(7), 2(11), 2(13), 7(11), 7(13), 11(13), 2(7)(11), 2(7)(13), 2(11)(13), 7(11)(13). The palindromes are 1, 2, 7, 11, 22, 77, 1001.
- B 9. Draw the diameter of the larger circle which passes through the center of the smaller circle. The required segment goes from the larger to the smaller circle along this diameter and has length $b - a - d$.
- A 10. The set can be rewritten as $\{24^{1000}, 28^{1000}, 32^{1000}, 27^{1000}, 25^{1000}\}$, so the first one is smallest.
- C 11. There is one ending 0 in $n!$ for every factor of 5 in every number $\leq n$, so the number of ending zeros for $n!$ is $n/5 + n/25 + n/125 + \dots$ (all rounded down). A little trial and error shows $n = 360$.
- A 12. Solving the equations $p + j + s = 60$ and $30p + 80j + 120s = 4400$ simultaneously yields $p = 8 + 0.8s$ and $j = 52 - 1.8s$. If $j \geq 0$, $s \leq 260/9$, so $s \leq 28$. But 25 is the largest value of s for which p is an integer.
- C 13. Since $\cos 3x \cos x - \sin 3x \sin x = \cos(3x + x)$, the equation is equivalent to $\cos 4x = 0$. Thus $4x$ is an odd multiple of $\pi/2$, and x is an odd multiple of $\pi/8$. The interval $[-8\pi, 8\pi]$ has 64 such multiples.
- D 14. If the triangle's legs are s and \sqrt{K} , then $s + \sqrt{K} + \sqrt{s^2 + K} = K$ and $(1/2)s\sqrt{K} = K$. But this implies $s = 2\sqrt{K}$. Substitution yields $3\sqrt{K} + \sqrt{5K} = K$, so $K = (3 + \sqrt{5})^2 = 14 + 6\sqrt{5}$, and $a + b + c = 25$.
- D 15. To leave a remainder of 1 on division by 2, 3, 4, 5, and 6, the number must be 1 more than a multiple of 60. The smallest such number divisible by 7 is 301, whose largest prime factor is 43.
- E 16. There are $10(9)(8) = 720$ ways to select 3 different digits, but only $1/6$ of them, or 120, have nondecreasing order. There are 90 ways to select 2 different digits, with half having nondecreasing order. The digits a, b produce 2 sequences, a, a, b or a, b, b , for 90 more. Finally, there are 10 sequences a, a, a .
- C 17. Since $AMA > TYC$ and $AMA - TYC < 10$, $A = T + 1$. Also, $M = 0$ and $Y = 9$ (otherwise, the difference ≥ 10). The uniqueness condition eliminates $AMA = 808, 707, 606$, and the only solutions are $505 - 498$ or $497, 404 - 398$ or $396, 303 - 298, 297, 296$, or 295 , and $202 - 198, 197, 195$, or 194 .
- B 18. $\triangle DCF$ is similar to each of the 3 small triangles formed by the intersection of the 2 rectangles. Solving a few similarity proportions shows the required region to be a trapezoid with height 12 and bases 40 and 15. Its area is thus $12(40 + 15)/2$ or 330.
- B 19. The required region is an isosceles right triangle with hypotenuse 20 ft, a three-quarter circle with radius 20 ft, and two 135° sectors with radii $20 - 10\sqrt{2}$ ft (forming another three-quarter circle). The total area is $\frac{1}{2}(10\sqrt{2})^2 + \frac{3}{4}(400\pi) + \frac{3}{4}\pi(20 - 10\sqrt{2})^2 = 100 + 750\pi - 300\pi\sqrt{2}$, or about 1123 sq ft.
- E 20. $P(\text{the numbers are different}) = 8/9$ (since for each first number, there are 8 different second numbers). $P(\text{the numbers are different if they add to } k)$ is 1, if k is odd (since two numbers with odd sum are never equal), and $(k-2)/(k-1)$ if k is even (since of the $k-1$ such sums, exactly one of them has two equal values). Independence means the two probabilities are equal, but this cannot happen if the second one is 1; thus k is even, and $(k-2)/(k-1) = 8/9$, meaning $k = 10$.